Oplossing oefening 90

Gegeven: 3 variabelen X_1, X_2, X_3

$Var(X_1) = Var(X_2) = Var(X_3) = \sigma^2$ (om het kind een naam te geven)

$\rho(X_1, X_2) = Corr(X_1, X_2) = 0.3$

$\rho(X_2, X_3) = Corr(X_2, X_3) = 0.2$

$\rho(X_1, X_3) = Corr(X_1, X_3) = 0.5$

$Y = X_1 + X_2$

$Z = X_2 + X_3$

Gevraagd: $\rho(Y, Z) = Corr(Y, Z)$

$$\rho(Y,Z) = Corr(X_1 + X_2, X_2 + X_3) = \frac{Cov(X_1 + X_2, X_2 + X_3)}{\sigma_{X_1+X_2}\sigma_{X_2+X_3}}$$

- $Cov(X_1 + X_2, X_2 + X_3)$

  Door 2 maal toepassen van eigenschap (5.33) p 80 vinden we:

  $$\begin{aligned} Cov(X_1 + X_2, X_2 + X_3) &= Cov(X_1, X_2 + X_3) + Cov(X_2, X_2 + X_3) \\ &= Cov(X_1, X_2) + Cov(X_1, X_3) + Cov(X_2, X_2) + Cov(X_2, X_3) \\ &= Cov(X_1, X_2) + Cov(X_1, X_3) + Var(X_2) + Cov(X_2, X_3) \end{aligned}$$

  Deze covarianties vinden we uit de opgave:

  $Cov(X_1, X_2) = Corr(X_1, X_2) \cdot \sigma_{X_1} \cdot \sigma_{X_2} = 0.3 \cdot \sigma \cdot \sigma = 0.3\sigma^2$

  $Cov(X_1, X_3) = Corr(X_1, X_3) \cdot \sigma_{X_1} \cdot \sigma_{X_3} = 0.5 \cdot \sigma \cdot \sigma = 0.5\sigma^2$

  $Cov(X_2, X_3) = Corr(X_2, X_3) \cdot \sigma_{X_2} \cdot \sigma_{X_3} = 0.2 \cdot \sigma \cdot \sigma = 0.2\sigma^2$

  Zodat:

  $Cov(X_1 + X_2, X_2 + X_3) = 0.3\sigma^2 + 0.5\sigma^2 + \sigma^2 + 0.2\sigma^2 = 2\sigma^2$

- $\sigma_{X_1+X_2} = \sqrt{Var(X_1 + X_2)}$

  Toepassen van eigenschap (5.30) (voor $a = b = 1, c = 0$) geeft:

  $$\begin{aligned} \sigma_{X_1+X_2} &= \sqrt{Var(X_1 + X_2)} \\ &= \sqrt{Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)} \\ &= \sqrt{\sigma^2 + \sigma^2 + 2 \cdot 0.3\sigma^2} \\ &= \sqrt{2.6}\sigma = \sqrt{\frac{13}{5}}\sigma \end{aligned}$$

- $$\begin{aligned} \sigma_{X_2+X_3} &= \sqrt{Var(X_2 + X_3)} \\ &= \sqrt{Var(X_2) + Var(X_3) + 2Cov(X_2, X_3)} \\ &= \sqrt{\sigma^2 + \sigma^2 + 2 \cdot 0.2\sigma^2} \\ &= \sqrt{2.4}\sigma = 2\sqrt{\frac{3}{5}}\sigma \end{aligned}$$

En dus is: $\rho(Y,Z) = \dfrac{\cancel{2}\,\cancel{\sigma}^2}{\sqrt{\frac{13}{5}}\cancel{\sigma} \cdot \cancel{2}\sqrt{\frac{3}{5}}\cancel{\sigma}} = \dfrac{5}{\sqrt{39}} = 0.80064$